MATHEMATICAL MODEL OF THE FREEZING–THAWING
OF SALINE FROZEN SOIL

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The need for a mathematical model of the interaction between frozen soils and salt solutions arises in studies of numerous natural and industrial processes. For instance, the interaction of seawater and northern subsea frozen rocks was shown [1–3] to affect significantly the dynamics of their thawing. The effect of salt solutions on frozen rocks during drilling with the use of salt-containing fluids was reported in [4, 5]. The presence of salt in the drilling fluid allows its cooling to negative temperatures, thereby decreasing the washout radius and, as a consequence, the cavity formation. The evolution of cryopegs (spaces in frozen rocks filled with salt-containing waters of negative temperatures) is another example of the natural process of interaction between frozen soils and salt solutions, given in [6].

In this paper, we developed a mathematical model of heat and mass transport involving phase transitions in frozen soils, obtained a self-similar solution (numerical and analytical for the linearized problem), and carried out a numerical treatment of the nonsteady-state problem.

1. The Model of Frozen Soil. A qualitative model of frozen soil based on the assumption that nonfreezing moisture is present in frozen soils was described in [7]: "... at any moment the frozen soil is assumed to contain some liquid water (filling the pores or interspersed between mineral particle contacts) in equilibrium with a number of ambient factors, such as temperature, pressure, and others." The equilibrium is not regarded as a merely thermodynamic one, but as a more general fact that frozen soils "respond" to environmental alterations, resulting in a new equilibrium state.

The H_2O phase transition temperature is known to depend on the concentration of the dissolved admixture, pressure, and the surface interaction forces. It is natural to assume that in pores of the frozen soil water exists in the liquid state at negative temperatures owing to the above-mentioned reasons and is in local thermodynamic equilibrium with pore ice. This thermodynamic equilibrium can be expressed quantitatively by the formula relating the temperature of the water–ice phase transition to the admixture concentration (the pressure and surface phenomena are beyond the scope of this article). In actual practice, use is more frequently made of the linear dependence

$$T_f = T_w - \alpha c, \tag{1.1}$$

where α is a coefficient characterizing the water–ice phase transition temperature reduction.

The thermodynamic equilibrium condition means that the local temperature of the frozen soil T is equal to the local temperature of the phase transition T_f calculated from (1.1). From this definition it is apparent that any environmental change upsets the thermodynamic equilibrium between ice and nonfreezing water within the pore space, resulting in either water crystallization or ice melting, accompanied by heat and mass transport until new equilibrium is achieved.

We consider a qualitative example. Let the aqueous salt solution in the pore space be in equilibrium with ice at a negative temperature $T_f = T_w - \alpha c$. Decreasing the frozen soil temperature by ΔT results in crystallization in pores, the salt being driven into the solution. This lasts until the admixture concentration increases by a value Δc such that a new equilibrium described by the equation $T_f - \Delta T = T_w - \alpha(c + \Delta c)$ is

achieved. Conversely, a rise in temperature results in ice melting and dilution of the solution, which reaches equilibrium with ice at an elevated temperature. Pressure changes act in a similar manner. To construct a mathematical model of phase transitions in frozen soils it is necessary to supplement the equilibrium condition (1.1) with transport equations and initial and boundary conditions, including those at mobile freezing–thawing interfaces.

2. Statement of the Problem. A mathematical description of heat and mass transport involving phase transformations in frozen soils can be derived within the framework of continuum mechanics from the laws of conservation of mass, momentum, and energy, and also from the laws of thermodynamics. Such an approach brings into existence a set of equations where the desired functions are pressure, concentration of dissolved admixtures, humidity, etc. We consider a set of equations describing variations in phase composition and transport of heat, water, and dissolved admixtures. Frozen soils are taken to be porous media saturated with ice and nonfreezing water. Gaseous components and water vapor are assumed absent. The matrix of the porous medium, ice, and pore fluid are incompressible and immobile. Admixture molecules are missing from the ice crystals. The whole of the dissolved admixture is driven into the solution and does not precipitate during ice formation. Ice and nonfreezing water are in thermodynamic equilibrium, that is, the characteristic time of the macroprocess is far beyond that of attainment of the local thermodynamic equilibrium.

In terms of these assumptions the law of conservation of mass, as applied to the dissolved matter, takes the form

$$\partial(m\,S\,c)/\partial t - \operatorname{div}(D\,m\,S\,\operatorname{grad} c) = 0\,, \tag{2.1}$$

where m is the porosity; S is the humidity; c is the mass concentration of the dissolved matter; D is the diffusion coefficient. This equation follows from the general continuity equation for dissolved matter and takes into account the diffusion mechanism of admixture transport and also the concentration (dilution) of the solution as the water saturation function decreases (increases). The convective mechanism of admixture transport is not considered here because of the assumption that no fluid flow takes place.

The energy equation accounting for heat liberation and absorption accompanying phase transitions in the frozen soil can be written as

$$(\rho\,C)_r\,\partial T/\partial t - \operatorname{div}(\lambda_r\,\operatorname{grad} T) + m\,\rho_i\,q\,\partial S/\partial t = 0. \tag{2.2}$$

Here

$$(\rho C)_r = (1-m)\rho_s C_s + mS\rho_w C_w + m(1-S)\rho_i C_i\,; \quad \lambda_r = (1-m)\lambda_s + mS\lambda_w + m(1-S)\lambda_i\,; \quad q = h_w - h_i\,;$$

ρ is the density, C is the heat capacity, λ is the heat conductivity, q is the heat of water–ice phase transition, and h is the enthalpy; the subscripts are s for soil matrix, w for water, and i for ice.

Statements of problems involving phase transitions include the desired moving phase-transition surfaces, where the function S can have a discontinuity. The conditions at the phase transition boundaries can be obtained from general relationships at discontinuity. We consider the jumps of the humidity function, at which the continuity conditions for temperature and admixture concentration are satisfied:

$$[T]_-^+ = 0, \quad [c]_-^+ = 0.$$

The laws of conservation of mass, dissolved admixture and energy at the jump take the form

$$m\,[S]_-^+\,c\,V_n + m\,[D\,S\,(\operatorname{grad} c)_n]_-^+ = 0\,; \tag{2.3}$$

$$m\,[S]_-^+\,q\,V_n + [\lambda_m\,(\operatorname{grad} T)_n]_-^+ = 0\,. \tag{2.4}$$

These conditions at humidity jumps must be supplemented by the thermodynamic equilibrium condition (1.1), yielding a complete set of conditions at unknown moving phase transition boundaries.

We consider the interaction between frozen soiles and a salt solution. Let a frozen soil in the equilibrium state at a temperature $T_0 = T_w - \alpha c_0$ and filling the half-space $x > 0$ be in contact with an aqueous salt solution with temperature T^0 and concentration c^0 ($T^0 > T_w - \alpha c^0$) at the immobile boundary $x = 0$. If the boundary temperature is lower than the initial one, then, taking into account the higher rate of heat

transport compared to admixture diffusion, it is reasonable to suggest that two fronts of phase transitions exist: the crystallization front initiated by cooling, and the melting front initiated by destruction of pore ice at its contact with the brine. As is shown in [8], the statement of a two-front problem contains a contradiction, which causes the violation of the thermodynamic equilibrium condition in the frozen soil zone. This is manifested as "overheating" or "overcooling" of the ice–brine mixture filling the pores. Thus, the frontal regime of phase transitions in frozen soils does not take place at all.

To construct a consistent theoretical description, it is sufficient to assume that phase transitions occur over the whole frozen soil zone. The fact that phase transitions propagate through the entire half-space is due to the temperature disturbances propagating at infinite velocity. In such a case, the mathematical statement of the problem assumes the existence of the front of partial phase transitions propagating to the right of $x = 0$ and separating the zones of thawed (left of the front) and frozen (ahead of the front) soil.

In the thawed soil zone the following equations for heat conductivity and salt diffusion hold true:

$$\partial T / \partial t = a_l \Delta T, \quad \partial c / \partial t = D \Delta c, \quad x < \chi(t) \tag{2.5}$$

$(a_l = [(1 - m)\lambda_s + m\lambda_w]/[(1 - m)\rho_s C_s + m\rho_w C_w])$.

Equations (2.1) and (2.2) is a system of transport equations involving phase transitions in the frozen soil zone.

At the moving boundary representing the thawing front of the frozen soil (ice absent at the back of the front) boundary conditions (2.3), (2.4) hold true.

If the disturbances of the desired functions are negligible compared to their absolute values, it is reasonable to consider the solution of the problem in a linear approximation. Thus, linearization of Eq. (2.1) near the initial state yields

$$S_0 \frac{\partial c}{\partial t} + c_0 \frac{\partial S}{\partial t} - D S_0 \Delta c = 0 \tag{2.6}$$

as a first approximation.

Note that the two remaining equations in the system are already linear. A system of equations for saturation and temperature disturbances is obtained by expressing the admixture concentration in terms of (1.1) and substituting it into (2.6):

$$\frac{\partial T}{\partial t} - \frac{\alpha c_0}{S_0} \frac{\partial S}{\partial t} = D \Delta T, \quad \frac{\partial T}{\partial t} + \frac{m q \rho_i}{(\rho C)_r} \frac{\partial S}{\partial t} = a_r \Delta T. \tag{2.7}$$

The humidity function appears only in the form of the time derivative. Having eliminated these derivatives, we obtain the effective heat equation

$$\frac{\partial T}{\partial t} = a_m \Delta T,$$

where

$$a_m = \frac{\lambda_m}{(\rho C)_m}; \quad \lambda_m = \frac{m D q S_0 \rho_i}{\alpha c_0 (\rho C)_r} + a_r; \quad (\rho C)_m = \frac{m q S_0 \rho_i}{\alpha c_0 (\rho C)_r} + 1.$$

3. Self-Similar Solution. We consider a solution of the linearized problem in the one-dimensional approximation. If T_0, c_0, S_0, T^0, and c^0 are constants, then the problem has a Neumann type self-similar solution

$$T = T(\xi), \quad c = c(\xi), \quad S = S(\xi), \quad x = \chi(t) = \varphi\sqrt{t}, \quad \xi = x/\sqrt{t}.$$

At $0 < x < \chi(t)$

$$T = T^0 + (T_* - T^0) \operatorname{erf}(\xi/2\sqrt{a_l})/\operatorname{erf}(\varphi/2\sqrt{a_l}), \quad c = c^0 + (c_* - c^0)\operatorname{erf}(\xi/2\sqrt{D})/\operatorname{erf}(\varphi/2\sqrt{D}); \tag{3.1}$$

at $x > \chi(t)$

$$T = T_0 + (T_* - T_0)\operatorname{erfc}(\xi/2\sqrt{a_m})/\operatorname{erfc}(\varphi/2\sqrt{a_m}). \tag{3.2}$$

The distribution of the admixture concentration in the region before the front is deduced from the equilibrium Eq. (1.1). To determine the humidity function in this domain, spatial derivatives are eliminated from system (2.7). This yields the following relationship between humidity and temperature:

$$T - T_0 = \sigma(S - S_0),$$

(3.3)

where

$$\sigma = \frac{D}{a_r}\left[\frac{m\,q\,\rho_i}{(\rho\,C)_r} + \frac{\alpha\,c_0\,a_r}{S_0\,D}\right].$$

The conditions at the moving boundary in the self-similar approximation take the form

$$m\,\rho_i\,q\,(1 - S_*)\frac{\varphi}{2} = \lambda_+\left(\frac{dT}{d\xi}\right)_+ - \lambda_-\left(\frac{dT}{d\xi}\right)_-,$$

$$\frac{c_*(1 - S_*)}{D}\frac{\varphi}{2} = S_*\left(\frac{dc}{d\xi}\right)_+ - \left(\frac{dc}{d\xi}\right)_-, \qquad T_* = T_w - \alpha c_*.$$

(3.4)

Substituting solutions (3.1)–(3.3) into (3.4) produces a system of transcendental equations for unknowns T_*, c_*, and φ:

$$m\,\rho_i\,q\,(1 - S)\frac{\varphi}{2} + \lambda_+\left(\frac{T_* - T_0}{\sqrt{\pi a_r}}\frac{\exp\left(-\varphi^2/4a_m\right)}{\operatorname{erfc}\left(\varphi/2\sqrt{a_m}\right)}\right)_+ + \lambda_-\left(\frac{T_* - T^0}{\sqrt{\pi\,a_l}}\frac{\exp(-\varphi^2/4a_l)}{\operatorname{erf}\left(\varphi/2\sqrt{a_l}\right)}\right)_- = 0,$$

$$\frac{c_*\,(1 - S_*)}{D}\frac{\varphi}{2} - \alpha\,S_*\left(\frac{T_* - T_0}{\sqrt{\pi a_r}}\frac{\exp\left(-\varphi^2/4a_m\right)}{\operatorname{erfc}\left(\varphi/2\sqrt{a_m}\right)}\right)_+ + \left(\frac{c_* - c^0}{\sqrt{\pi D}}\frac{\exp\left(-\varphi^2/4D\right)}{\operatorname{erf}\left(\varphi/2\sqrt{D}\right)}\right)_- = 0, \quad c_* = \frac{T_w - T_*}{\alpha}.$$

If the disturbances of the desired functions are comparable to their absolute values, the construction of a self-similar solution requires solving a two-point boundary-value problem for the system of ordinary differential equations for the frozen zone, obtained from (2.1), (2.2) by the replacement $\xi = x/\sqrt{t}$, and also a system of transcendental equations for the required interface, temperature, and concentration at this boundary.

4. Numerical Solution for the Nonsteady-State Problem. For numerical solution of the nonclassical problem considered here, we use an implicit finite-difference scheme combining advantages of the finite-difference predictor–corrector method and of the approach of fixing the ice–two-phase zone interface at a node of a rectangular spatial-temporal grid [9, 10]. On the segment $[0, 1]$ we introduce a quasi-uniform grid

$$\overline{\omega}_h = \left\{\,x_i = x_{i-1} + h_i, \quad i = \overline{1, n};\quad x_0 = 0;\quad h_i = a\,h_{i-1}, \quad i = \overline{1, n}\,\right\},$$

whose steps form an increasing geometrical progression with a denominator $a > 1$. The grid parameters n, h_0, a are chosen so that $x_n = L$. The magnitude of the time step is determined during the solution of the problem.

We assume that approximate values of the solution to the original problem are known up to the instant $t = t_{j-1}$. A computational algorithm for the transition into the next time layer can now be constructed.

The implicit discrete analogs of the heat transport and diffusion equations in the thawed zone (2.5) are written

$$\hbar_i\frac{T_i - \check{T}_i}{a_l\tau} = \frac{T_{i+1} - T_i}{h_{i+1}} - \frac{T_i - T_{i-1}}{h_i}, \quad i = \overline{1, j - 1};$$

(4.1)

$$\hbar_i\frac{c_i - \check{c}_i}{D\,\tau} = \frac{c_{i+1} - c_i}{h_{i+1}} - \frac{c_i - c_{i-1}}{h_i}, \quad i = \overline{1, j - 1}.$$

(4.2)

From Eqs. (3.1), (2.2) and relationship (1.1) we deduce an equation from which the temperature distribution in the frozen zone is to be found. From (2.1) by virtue of (1.1) it follows that

$$(T - T_w)\frac{\partial S}{\partial t} = D\frac{\partial}{\partial x}\left(S\frac{\partial T}{\partial x}\right) - S\frac{\partial T}{\partial t}, \quad x > \chi(t), \quad t > 0.$$

Substituting the right-hand side of this relationship for the last term into the right-hand side of differential equation (2.2) yields

$$(\rho C)_r \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_r \frac{\partial T}{\partial x}\right) - m\,\rho_i\,q\,\frac{D}{T - T_w}\frac{\partial}{\partial x}\left(S\frac{\partial T}{\partial x}\right) + m\,\rho_i\,q\,\frac{1}{T - T_w}\frac{\partial T}{\partial t}\,, \quad x > \chi(t)\,, \quad t > 0\,.$$

We approximate this equation by implicit three-point finite–difference equations:

$$\hbar_i\left[(\rho C)_{r,i} - \frac{\rho_i\,q\,m}{T_i - T_w}\right]\frac{T_i - \check{T}_i}{\tau} = \left[\lambda_{r,i+1/2} - \frac{m\,q\,\rho_i\,D}{T_i - T_w}\right]\frac{T_{i+1} - T_i}{h_{i+1}} - \left[\lambda_{r,i-1/2} - \frac{m\,q\,\rho_i\,D}{T_i - T_w}\right]\frac{T_i - T_{i-1}}{h_i}\,,$$

$$i = \overline{j+1, n-1}\,.$$

(4.3)

Here

$$(\rho C)_{r,k} = (1-m)\,\rho_s\,C_s + m\,S_k\,\rho_w\,C_w + m\,(1-S_k)\,\rho_i\,C_i;$$

$$\lambda_{r,k+1/2} = (1-m)\lambda_s + m\,\lambda_i + m(\lambda_w - \lambda_i)(S_k + S_{k+1})/2\,.$$

The distribution of the salt concentration in the frozen zone is found from a discrete analog of relationship (1.1) expressing the thermodynamic equilibrium of water and ice:

$$c_i = (T_w - T_i)/\alpha\,, \quad i = \overline{j, n}\,.$$

(4.4)

By means of the additivity principle [9], the Roth differential–finite-difference scheme

$$(\rho C)_r \frac{T - \check{T}}{\tau} = \frac{\partial}{\partial x}\left(\lambda_r \frac{\partial T}{\partial x}\right) - \rho_i\,q\,m\,\frac{\partial S}{\partial t}\,, \quad x > \chi(t)\,,$$

approximating the law of conservation of energy can be carried out numerically in two steps. In the first step, an auxiliary temperature distribution is found as a solution of the following heat equation:

$$(\rho C)_r \frac{\overline{T} - \check{T}}{\tau} = \frac{\partial}{\partial x}\left(\lambda_p \frac{\partial \overline{T}}{\partial x}\right)\,, \quad x > \chi_j$$

$[\lambda_p = (1-m)\lambda_s + m\lambda_i]$. We write its implicit discrete analog

$$\hbar_i\,(\rho C)_{r,i}\frac{\overline{T}_i - \check{T}_i}{\lambda_p\,\tau} = \frac{\overline{T}_{i+1} - \overline{T}_i}{h_{i+1}} - \frac{\overline{T}_i - \overline{T}_{i-1}}{h_i}\,, \quad i = \overline{j+1, n-1}\,.$$

(4.5)

In the second step, the Roth scheme for the temperature distribution in the upper time layer is written

$$(\rho C)_r \frac{T - \overline{T}}{\tau} = (\lambda_l - \lambda_p)\frac{\partial}{\partial x}\left(S\frac{\partial T}{\partial x}\right) - \rho_i\,q\,m\,\frac{\partial S}{\partial t}\,, \quad x > \chi_j\,.$$

Since ice and water in the two-phase zone are in thermodynamic equilibrium, and local values of temperatures and of the salt concentration in the solution satisfy condition (1.1), from the relationship deduced above, and using Eq. (2.1) we obtain

$$(\rho C)_r \frac{T - \overline{T}}{\tau} = (\lambda_p - \lambda_l)\frac{S(T - T_p) + \alpha\,\check{S}\,\check{c}}{D\,\tau} - \rho_i\,q\,m\,\frac{S - \check{S}}{\tau}\,, \quad x > \chi_j\,.$$

A discrete analog of this equation gives the humidity distribution in the two-phase zone:

$$S_i = \frac{\rho_i\,q\,m\,\check{S}_i - (\rho C)_p\,(T_i - \overline{T}_i) + (\lambda_p - \lambda_l)\,\alpha\,\check{S}_i\,\check{c}_i}{\rho_i\,q\,m + [(\rho C)_l - (\rho C)_p](T_i - \overline{T}_i) - (\lambda_p - \lambda_l)(T_i - T_w)}\,, \quad i = \overline{j, n-1}$$

(4.6)

$[(\rho C)_p = (1-m)(\rho C)_s + m(\rho C)_i]$.

Prior to constructing discrete analogs of the boundary conditions at the unknown interface between the thawed and frozen zones we exclude the term containing the velocity of interfacial motion from relationship (2.3), using condition (2.4):

$$(T_* - T_w)\left[\lambda \frac{\partial T}{\partial x}\right] = \alpha\,q\,\rho_i\,D\left[S\frac{\partial c}{\partial x}\right]\,, \quad t > 0\,.$$
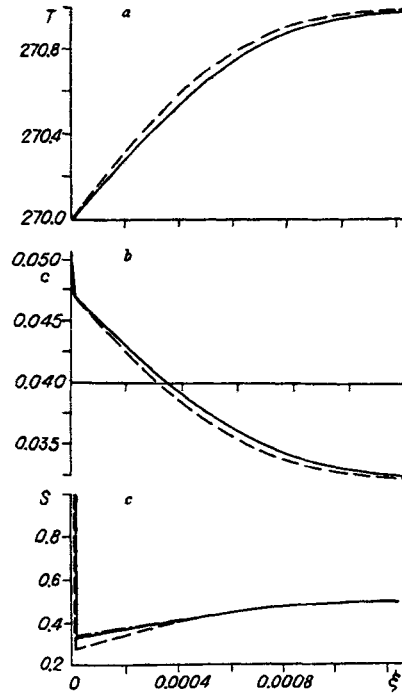
Fig. 1

We write a discrete analog of this relationship:

$$(T_j - T_w) \left[\lambda_{r,j} \frac{T_{j+1} - T_j}{h_{j+1}} - \lambda_l \frac{T_j - T_{j-1}}{h_j} \right] = m \alpha q \rho_i D \left[\frac{c_{j+1} - c_j}{h_{j+1}} S_j - \frac{c_j - c_{j-1}}{h_j} \right], \qquad (4.7)$$

and an implicit finite-difference approximation of the generalized Stefan condition (2.4):

$$m q \rho_s (1 - S_j) \frac{h_j}{\tau} = \lambda_{r,j} \frac{T_{j+1} - T_j}{h_{j+1}} - \lambda_l \frac{T_j - T_{j-1}}{h_j}. \qquad (4.8)$$

Discrete analogs of the boundary conditions at fixed boundaries are constructed in a similar manner.

Hence, for the transition from the time layer $(j - 1)$ to the next layer j a nonlinear system of algebraic equations should be solved. The nonlinearity of the system considered stems from the fact that we should know the time step in order to find the distributions of temperature, of the admixture concentration, and of humidity. Conversely, the calculation of the time step requires the knowledge of the temperature distribution or, at least, of temperature values in the vicinity of the interface between the thawed and frozen zones. Therefore, the solution of the resulting nonlinear system requires an iterative process. We use the following successive approximation method, which is essentially a variation of the method of ordinary iterations.

Step 1. We set $k = 0$ (the iteration counter) and specify the initial approximation for the time step τ_0 (for instance, $\tau_0 = \bar{\tau}$). Clearly, it is necessary to set a separate initial approximation for the first time step, for example, by the formula

$$\tau_0 = \sigma q \rho_i h_1^2 / [\lambda_l (T_w - \alpha c_0 - T_c)], \quad \sigma \in (0, 1].$$

Step 2. We solve the linearized system of Eqs. (4.1)–(4.6) with the corresponding boundary conditions at the fixed boundaries.

Step 3. We determine the next approximation for the phase transition temperature T_j from the linearized Eq. (4.7) and the time step from relationship (4.8). In doing so the least of the roots of the resulting square equation is taken as the next approximation for the ice–water phase transition temperature. The admixture concentration at the front can be deduced from the linearized relationship (4.4)
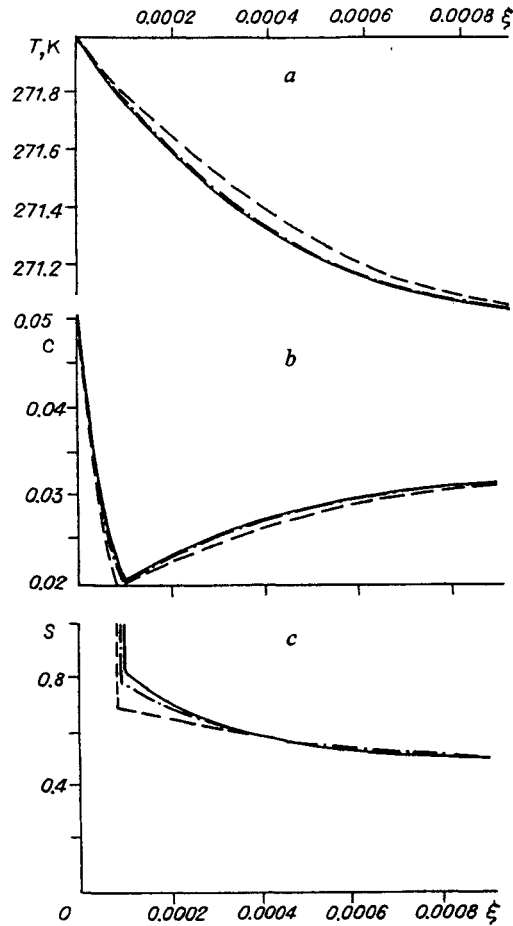
Fig. 2

Step 4. We consider the iterative process error

$$z = |\tau_{k+1}/\tau_k - 1|.$$

If $z \geqslant \varepsilon$, where ε is a small preset value, we increase k by one and return to step 2 in the iterative process. Otherwise the transition to the next temporal layer is assumed to be accomplished, and the desired time step, temperature distribution, and admixture and moisture concentrations in both phases and at the interface take the corresponding values from the last iteration.

5. Calculation Examples. Below are given the computation results for three solutions to the one-dimensional problem to a linearized self-similar problem, to a self-similar problem in its original formulation, and a numerical solution to a nonsteady-state problem. The computation was made at the following parameters: $\rho_w = 1{,}000$ kg/m^3, $\rho_i = 910$ kg/m^3, $\rho_s = 2{,}000$ kg/m^3, $m = 0.2$, $\lambda_w = 0.58$ W/(m·K), $\lambda_i = 2.23$ W/(m·K), $\lambda_s = 2$ W/(m·K), $c_w = 4{,}190$ J/(kg·K), $c_i = 2{,}000$ J/(kg·K), $c_s = 1{,}920$ J/(kg·K), $q = 3.34 \cdot 10^5$ J/kg, $D = 1.45 \cdot 10^{-9}$ m^2/sec, $\alpha = 66.7$ K, $T_w = 273.15$ K, $S_0 = 0.5$, $T_0 = 271$ K, $c^0 = 0.05$.

Figures 1 and 2 show the characteristic distributions of the desired functions in both zones. The solid line represent the numerical solution to the nonsteady-state problem; the dashed line, the numerical solution to the self-similar problem; and the dotted-dashed line, the solution to the linearized problem. The results of numerical experiments demonstrate the existence of two essentially different regimes of the interaction between frozen soils and salt solutions.

In the first case (Fig. 1), where the temperature of the salt solution ($T^0 = 272$) is higher than that of the frozen soil, ice melting in the domain before the front can be observed along with the front of complete

thawing. The given distribution of salt concentration indicates the dilution of the solution in the vicinity of the front owing to additional water resulting from ice melting. The decrease in admixture concentration causes its redistribution in both zones. This factor, along with heat transport, results in violation of thermodynamic equilibrium in the frozen soil zone, thereby causing phase transitions in an extended region before the front.

The second case (Fig. 2) occurs when frozen soils interact with a high-salt solution whose temperature $(T^0 = 270)$ is lower than that of the frozen soil. Here the existence of the complete thawing front is due to the salt "corroding" pore ice. This process is accompanied by a considerable heat absorption at the thawing front, and the temperature at the front becomes lower than the initial temperatures of the frozen soil and the salt solution. The cooling results in heat removal from the frozen soil zone, causing partial freezing. This is evidenced by a decrease in the humidity function at the front and in the region before it, as compared to its initial value. The propagation rate of the complete thawing front is determined mainly by the magnitude of heat flow from the thawed soil zone and is much higher in the case of the warmer solution (Fig. 1). It is noteworthy that the solutions to the precise and linearized formulations of the problem are satisfactorily similar.

The constructed mathematical model for frozen soil and phase transitions therein provides an explanation for some phenomena occurring during drilling of frozen rocks with the use of aqueous fluids when thawing and rock entrainment are much more pronounced than could be expected on the basis of the front heat problem. Indeed, if a frozen rock contains nonfreezing water with even a small amount of admixture, and the solution temperature is higher than that of the frozen soil, then thawing proceeds as is qualitatively shown in Fig. 1. That is, not only completely thawed zone emerges, as in the front model, but considerable partial thawing also takes place. Since the durability of frozen rocks is largely determined by their humidity (ice strengthens the rock), the achievement of a critical humidity during partial thawing will lead to a collapse and carrying-away of such incompletely thawed rocks. On the other hand, when using salt-containing cooled solutions for soil freezing, it should be taken into account that the freezing is accompanied by a slower process of thawing due to ice "corrosion" caused by the salt solution (Fig. 2).

Similar phenomena are involved in the interaction between artificial structures and frozen soils. This case can be modeled by a wall permeable to heat but not to admixture. Then heat, coming into the soil from a warmer surface, will cause thawing in an extended zone, resulting in deterioration in durability of a region much larger than that obtained from the Stefan heat problem.

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